Binomial Theorem

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \left({}^nC_k \, a^{(n-k)} b^k \right)}$$

$$\Rightarrow \quad (a+b)^n = {}^nC_0 \, a^{(n-0)} b^0 + {}^nC_1 \, a^{(n-1)} b^1 + \ldots + {}^nC_n \, a^{(n-n)} b^n .$$

$$\boxed{(\text{Total no. of terms}) = (n+1)}$$

General Term – $\boxed{T_{(r+1)} = {}^nC_r \, a^{(n-r)} \, b^r}$

$(r+1)$th term.

Props of ${}^nC_r$

1) ${}^nC_r = {}^nC_{(n-r)}$

2) ${}^nC_r = {}^nC_k \Leftrightarrow r=k$ or $(r+k)=n$

3) ${}^nC_r + {}^nC_{(r+1)} = {}^{(n+1)}C_{(r+1)}$

4) $\left(\dfrac{{}^nC_r}{{}^nC_{(r-1)}}\right) = \dfrac{(n-r+1)}{(r)}$

5) $n =$ even, ${}^nC_r$ greatest if $r = n/2$

$n =$ odd, ${}^nC_r$ greatest if $r = (n+1)/2, (n-1)/2$

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Middle Term

1) If $n =$ even, $\left(\text{Middle Term}\right) = \left(\frac{n}{2} + 1\right)$th Term.

2) If $n =$ odd, $\left(\text{Middle Term}\right) = \left[\left(\frac{n+1}{2}\right)\text{th Term}\right]$ & $\left[\left(\frac{n+3}{2}\right)\text{th Term}\right]$

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Numerically Greatest Term

If $T_{(r+1)}$ is numerically greatest then,

$$\left|\frac{T_{(r+1)}}{T_{(r)}}\right| \geq 1 \quad \& \quad \left|\frac{T_{(r+1)}}{T_{(r+2)}}\right| \geq 1$$

$$\Rightarrow \left|\frac{n+1-r}{r}\right|\left|\frac{b}{a}\right| \geq 1 \quad \& \quad \left|\frac{r+1}{n-r}\right|\left|\frac{a}{b}\right| \geq 1$$

$$\Rightarrow \left(\frac{n+1}{r} - 1\right) \geq \left|\frac{a}{b}\right| \quad \& \quad (r+1) \geq \left|\frac{b}{a}\right|(n-r)$$

$$\Rightarrow \boxed{r \leq \left(\frac{(n+1)}{1+|a/b|}\right)} \quad \& \quad r \geq \left(\frac{n|b/a| - 1}{1+|b/a|}\right)$$

$$\Rightarrow \boxed{r \geq \left(\frac{n+1}{1+|a/b|}\right) - 1}$$

To find Numerically Greatest Term in expansion of $\boxed{(a+b)^n}$

calc. the qty $\boxed{\frac{n+1}{1+|a/b|}}$

Let $m = \left[\left(\frac{n+1}{1+|a/b|}\right)\right]$. If —

1) $m \notin \mathbb{Z} \Rightarrow$ Numerically Greatest Term is $T_{(m+1)}$

2) $m \in \mathbb{Z} \Rightarrow$ Numerically Greatest Terms are T_m & T_{m+1}

Greatest Binomial Coeficient —

We can use above theory in expansion of $(1+1)^n$

as $(1+1)^n = {}^nC_0 + {}^nC_1 + \dots + {}^nC_{(n-1)} + {}^nC_n$

Here, $m = \left(\frac{n+1}{2}\right)$.

$$\Rightarrow \left(\begin{array}{c}\text{Greatest Binomial} \\ \text{Coeff.}\end{array}\right) = \begin{cases} {}^nC_{n/2}, & n = \text{even} \\ {}^nC_{\left(\frac{n-1}{2}\right)} \text{ \& } {}^nC_{\left(\frac{n+1}{2}\right)}, & n = \text{odd} \end{cases}$$

Q) Find greatest term in $(2+3x)^9$ if $x = 3/2$.

A) $m = \dfrac{(9+1)}{1+\left|\frac{2}{3x}\right|} \Rightarrow m = \dfrac{9(\cancel{9}+1)}{(13/9)} \Rightarrow m = \left(\dfrac{90}{13}\right) = 6.\dots$

$\Rightarrow T_7$ is greatest

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Q) Find value of 'r' for which $^{25}C_r$ attains its max. value.

A) Since $25 = \text{odd}$, $^{25}C_r$ is max. is $r = \left(\frac{25 \pm 1}{2}\right)$

$$\Rightarrow \boxed{r = 12, 13}$$

Binomial Series

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 + \dots + (-1)^n C_n x^n$$

Combining above eqⁿs, using $x = 1$,

1) $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

2) $C_0 - C_1 + C_2 + \dots + (-1)^n C_n = 0$

3) $C_0 + C_2 + C_4 + C_6 + \dots = 2^{(n-1)}$

4) $C_1 + C_3 + C_5 + C_7 + \dots = 2^{(n-1)}$

Eg – P.t. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{(n-1)}$

M1: Calculus Method.

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$$

$$\xrightarrow{d/dx} \; n(1+x)^{(n-1)} = C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{(n-1)}$$

Let $x=1$, $\boxed{C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{(n-1)}}$

M2: General Term Method.

$$\boxed{\text{Req.}} = \sum_{r=0}^{n}\left(r \cdot {}^{n}C_r\right) = \sum_{r=0}^{n}\left(r \cdot \frac{\lfloor n}{\lfloor r \cdot \lfloor n-r}\right)$$

$$= \sum_{r=1}^{n}\left(n \cdot \frac{\lfloor n-1}{\lfloor r-1 \cdot \lfloor n-r}\right) = n\left(\sum_{r=1}^{n}\left({}^{(n-1)}C_{(r-1)}\right)\right)$$

$$= n\left({}^{(n-1)}C_0 + {}^{(n-1)}C_1 + \ldots + {}^{(n-1)}C_{(n-1)}\right) = \boxed{n \cdot 2^{(n-1)}}$$

M3: A.P. Method

$$\text{Req} = [C_1 + 2C_2 + \ldots + (n-1)C_{(n-1)}] + nC_n \qquad \left({}^{n}C_r = {}^{n}C_{(n-r)}\right)$$

$$\text{Req} = [(n-1)C_1 + (n-2)C_2 + \ldots + 1 \cdot C_{(n-1)}] + nC_n$$

$$2(\text{Req.}) = (n)\left({}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_{(n-1)}\right) + 2n \Rightarrow \boxed{\text{Req.} = n \cdot 2^{(n-1)}}$$

Eg – P.t. $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{(n+1)} = \left(\frac{2^{(n+1)}-1}{(n+1)}\right)$

M1: Calculus Method.

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n \cdot x^n$$

$$\cancel{\left(\int dx\right)} \Rightarrow \left[\frac{(1+x)^{(n+1)}}{(n+1)}\right]_0^1 = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \dots + \frac{C_n x^{(n+1)}}{(n+1)}\right]_0^1$$

$$\Rightarrow \boxed{\left(\frac{2^{(n+1)}-1}{n+1}\right) = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{(n+1)}}$$

M2: General Term Method.

$$\boxed{\text{Req.}} = \sum_{r=0}^{n}\left(\frac{{}^nC_r}{(r+1)}\right) = \sum_{r=0}^{n}\left(\frac{\lfloor n}{(r+1)\lfloor r \lfloor n-r}\right) = \sum_{r=0}^{n}\left(\frac{\lfloor n}{\lfloor r+1 \lfloor n-r}\right)$$

$$= \sum_{r=0}^{n}\left(\frac{1}{(n+1)} \cdot \frac{\lfloor n+1}{\lfloor r+1 \lfloor n-r}\right) = \sum_{r=0}^{n}\left(\frac{{}^{(n+1)}C_{(r+1)}}{n+1}\right)$$

$$= \frac{1}{(n+1)}\left({}^{(n+1)}C_1 + {}^{(n+1)}C_2 + \dots + {}^{(n+1)}C_{(n+1)}\right) = \boxed{\frac{2^{(n+1)}-1}{(n+1)}}$$

Product of Coeff. Series

☆ Q) $C_0 C_r + C_1 C_{(r+1)} + C_2 C_{(r+2)} + \dots + C_{(n-r)} C_n = \left(\dfrac{\lfloor 2n}{\lfloor n-r \; \lfloor n+r} \right)$

A) Consider it as finding ~~coeff~~ coeff. of $\left(x^{(n+r)}\right)$ in a certain series.

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$$C_0 C_r + C_1 C_{(r+1)} + C_2 C_{(r+2)} + \dots + C_{(n-r)} C_n$$

$$\Updownarrow$$

$$\underbrace{C_n C_r}_{(n+r)} + \underbrace{C_{(n-1)} C_{(r+1)}}_{(n+r)} + \underbrace{C_{(n-2)} C_{(r+2)}}_{(n+r)} + \dots + \underbrace{C_r C_n}_{(n+r)}$$

Sum:

☆ Make sum of subscripts a const.

Now, $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + \dots + {}^nC_n x^n$

$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + \dots + {}^nC_n x^n$.

Let us multiply term $T_{(a+1)}$ & $T_{(b+1)}$ from 1st & 2nd exp^n. resp.

$\Rightarrow \quad {}^nC_a \cdot x^a \cdot {}^nC_b \cdot x^b = {}^nC_a \, {}^nC_b \cdot x^{(a+b)}$

If $(a+b) = (n+r) \Rightarrow$ Req. series becomes coef. of $x^{(n+r)}$ in $(1+x)^{2n}$

$\Rightarrow \quad \boxed{\text{Series} = {}^{2n}C_{(n+r)}}$

Q) ~~Find~~ P.t. $C_0^2 + C_1^2 + \dots + C_n^2 = {}^{2n}C_n$.

A) In prev. Q, take $r=0$.

Q) P.T. $C_0^2 - C_1^2 + C_2^2 + \dots + (-1)^n C_n^2 = \begin{cases} 0 & ; \ n = \text{odd} \\ (-1)^{n/2}\, {}^nC_{(n/2)} & ; \ n = \text{even} \end{cases}$

A) $\text{Req.} \equiv C_0 C_n - C_1 C_{(n-1)} + C_2 C_{(n-2)} + \dots + (-1)^n C_n \cdot C_0$

$$(1+x)^n = C_0 x^0 + C_1 x^1 + \dots + C_n x^n$$

$$(1-x)^n = C_0 x^0 - C_1 x^1 + \dots + C_n (-1)^n \cdot x^n$$

Req. is coeff. of x^n in $(1-x^2)^n$.

If $n = \text{odd}$, coeff. $= 0$ (only even powers in expⁿ!)

If $n = \text{even}$, coeff. $= {}^nC_{(n/2)} \cdot (-1)^{n/2}$ (as we need coeff. of $(x^2)^{(n/2)}$)

Hence,

$$\text{Req.} = \begin{cases} 0 & ; \ n = \text{odd} \\ (-1)^{n/2}\, {}^nC_{(n/2)} & ; \ n = \text{even} \end{cases}$$

★ In Q on Pg. 33, we could have multiplied the series

$(1+x)^n$ & $(1+1/x)^n$

& found coeff. of (x^r) in the product!

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Q) $^mC_r + {^mC_{r-1}}\,{^nC_1} + {^mC_{r-2}}\,{^nC_2} + \dots + {^nC_r} = {^{(m+n)}C_r}$; $r \leq \min\{m,n\}$

A) finding coeff. of x^r in $(1+x)^m(1+x)^n$

$$\Rightarrow \boxed{^{(m+n)}C_r}$$

$$(1+x)^m = {^mC_0}x^0 + {^mC_1}x^1 + \dots + {^mC_m}x^m$$

$$(1+x)^n = {^nC_0}x^0 + {^nC_1}x^1 + \dots + {^nC_n}x^n$$

Select x^a from $(1+x)^m$ & $x^{(r-a)}$ from $(1+x)^n$.

$$\Rightarrow \text{Coeff. of } x^r = \sum_{a=0}^{r} {^mC_a}\,{^nC_{(r-a)}} = \text{Req.}$$

☆ Q) $^nC_0\,{^{2n}C_n} - {^nC_1}\cdot{^{(2n-1)}C_n} + {^nC_2}\cdot{^{(2n-2)}C_n} + \dots + (-1)^n\,{^nC_n}\,{^nC_n} = ?$

A) Req. = Coeff. of x^n in $\left[{^nC_0}(1+x)^{2n} - {^nC_1}(1+x)^{(2n-1)} + {^nC_2}(1+x)^{(2n-2)} - \dots + (-1)^n\,{^nC_n}(1+x)^n \right]$

$\Rightarrow$ Coeff. of x^n in $(1+x)^n\left[{^nC_0}(1+x)^n - {^nC_1}(1+x)^{n-1} + \dots + (-1)^n\,{^nC_n}\right]$

$\Rightarrow$ Coeff. of x^n in $(1+x)^n\left[(1+x)-1\right]^n$

$\Rightarrow$ Coeff. of x^n in $(1+x)^n \cdot x^n$

$\Rightarrow$ Coeff. of x^0 in $(1+x)^n = \boxed{1}$

Q1) If $n \in \mathbb{Z}^+$ & $(1+x+x^2)^n = \sum_{r=0}^{2n}(a_r x^r)$, then p.t —

1) $a_r = a_{(2n-r)}$; $0 \leqslant r \leqslant 2n$

2) $a_0 + a_1 + \dots + a_{(n-1)} = \frac{1}{2}(3^n - a_n)$

3) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + (-1)^{n-1} a_{(n-1)}^2 = \frac{1}{2}(a_n)(1-(-1)^n a_n)$

A) 1) $(1+x+x^2)^n = \sum_{r=0}^{2n}(a_r x^r)$ —— (1)

$\Rightarrow \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n}\left(a_r\left(\frac{1}{x}\right)^r\right)$ $\{x \to 1/x\}$

$\Rightarrow x^{2n}\left(\frac{1+x+x^2}{x^2}\right)^n = \sum_{r=0}^{2n}\left(a_r \cdot x^{(2n-r)}\right) = \sum_{k=0}^{2n}\left(a_{(2n-k)} \cdot x^k\right)$

$\{k = (2n-r)\}$

$\Rightarrow (1+x+x^2)^n = \sum_{k=0}^{2n}\left(a_{(2n-k)} \cdot x^k\right)$ —— (2)

Comparing (1) & (2), $\boxed{a_r = a_{(2n-r)}}$ {k & r are dummy variables}

2) Put $x=1$, $3^n = (a_1 + a_2 + \dots + a_{n-1}) + a_n + (a_{n+1} + \dots + a_{2n})$

$= 2(a_1 + a_2 + \dots + a_{n-1}) + a_n$

$\Rightarrow \boxed{[a_1 + a_2 + \dots + a_{(n-1)}] = \frac{1}{2}(3^n - a_n)}$

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3) $(1+x+x^2)^{2n} = \sum_{r=0}^{2n}\left(a_r \cdot x^r\right)$

$= \sum_{r=0}^{2n}\left(a_r \cdot x^{(2n-r)}\right)$

Let $x \rightarrow (-x)$, $(1-x+x^2)^{2n} = \sum_{r=0}^{2n}\left(a_r \cdot (-1)^r \cdot x^r\right)$

& $(1+x+x^2)^{2n} = \sum_{r=0}^{2n}\left(a_r \cdot x^{(2n-r)}\right)$

$\therefore$ We need coeff. of x^{2n} in $(1-x+x^2)^{2n}(1+x+x^2)^{2n}$

$\Rightarrow$ " " x^{2n} " $\left((x^2+1)^2 - x^2\right)^{2n}$

$\Rightarrow$ " " x^{2n} " $\left(x^4+x^2+1\right)^{2n}$

Let $x \rightarrow \sqrt{x}$, " " x^n " $(1+x+x^2)^{2n}$

$= a_n$

Hence $\left(a_0^2 - a_1^2 + \ldots\right) + (-1)^n a_n^2 + \left((-1)^{n+1} a_{n+1}^2 + \ldots + (-1)^{2n} a_{2n}^2\right) = a_n$

$\Rightarrow 2\left(a_0^2 - a_1^2 + \ldots + (-1)^{(n-1)} a_{(n-1)}^2\right) + (-1)^n a_n^2 = a_n$

$\Rightarrow \boxed{a_0^2 - a_1^2 + \ldots + (-1)^{(n-1)} a_{(n-1)}^2 = \frac{1}{2}(a_n)\left(1-(-1)^n a_n\right)}$

Q) If $p+q=1$, then find the following in terms of n, p, q.

1) $\sum_{r=0}^{n}\left({}^{n}C_r\, p^{r}\, q^{(n-r)}\right)$ 2) $\sum_{r=0}^{n}\left(r\cdot{}^{n}C_r\cdot p^{r}\cdot q^{(n-r)}\right)$

3) $\left[\sum_{r=0}^{n}\left(r\cdot{}^{n}C_r\, p^{r}\, q^{(n-r)}\right)\right]^2-\left[\sum_{r=0}^{n}\left(r^2\cdot{}^{n}C_r\cdot p^{r}\cdot q^{(n-r)}\right)\right]$

A) 1) $(p+q)^n = 1$

2) $\sum_{r=0}^{n}\left(r\cdot\frac{\lfloor n}{\lfloor r\cdot\lfloor n-r}\cdot p^{r}\cdot q^{(n-r)}\right) = (np)\left(\sum_{r=1}^{n}\left({}^{(n-1)}C_{(r-1)}\cdot p^{(r-1)}\cdot q^{(n-r)}\right)\right)$

$= (np)(p+q)^{(n-1)} = np$

3) $\sum_{r=0}^{n}\left(r\cdot{}^{n}C_r\cdot p^{r}\, q^{(n-r)}\right) = np$ —— ①

$\sum_{r=0}^{n}\left(r^2\cdot{}^{n}C_r\cdot p^{r}\cdot q^{(n-r)}\right) = \sum_{r=1}^{n}\left(nr\cdot{}^{(n-1)}C_{(r-1)}\cdot p^{r}\, q^{(n-r)}\right)$

$=(np)\left(\sum_{r=1}^{n}\left({}^{(n-1)}C_{(r-1)}\cdot p^{(r-1)}\cdot q^{(n-r)}\right)\right) + n(n-1)\left(\sum_{r=2}^{n}\left({}^{(n-2)}C_{(r-2)}\cdot p^{(r-2)}\cdot q^{(n-r)}\right)\right)\cdot p^2$

$= (np)(p+q)^{(n-1)} + n(n-1)p^2(p+q)^{(n-2)} = np + n(n-1)p^2$ —— (

$$\text{Req} = \left(1^2 - 2\right) = n^2p^2 - np - n(n-1)p^2$$

$$= \cancel{n^2p^2} - np - \cancel{n^2p^2} + np^2 = np(p-1)$$

$$= -npq$$

Alternate –

2) Diff. $(px+q)^n$ wrt. 'x'.

3) Diff. $(px+q)^n$ wrt. 'x' to get first term

Diff. $(px+q)^n$, multiply 'x' & diff. again to get second term.

Q) P.T. $$\sum_{r=0}^{n}\left[\frac{{}^nC_r \cdot 3^{(r+4)}}{(r+1)(r+2)(r+3)(r+4)}\right] = \left[\frac{4^{(n+4)} - \sum_{r=0}^{3}\left({}^{(n+4)}C_r \cdot 3^r\right)}{(n+1)(n+2)(n+3)(n+4)}\right]$$

A) $$\frac{{}^nC_r}{(r+1)(r+2)(r+3)(r+4)} = \frac{\lfloor n}{\lfloor r \cdot \lfloor n-r} \cdot \frac{1}{(r+1)(r+2)(r+3)(r+4)}$$

$$= \frac{\lfloor n}{\lfloor r+4 \, \lfloor n-r} = \frac{\lfloor n+4}{\lfloor r+4 \, \lfloor n-r} \cdot \frac{1}{(n+1)(n+2)(n+3)(n+4)} = \frac{{}^{(n+4)}C_{(r+4)}}{(n+1)(n+2)(n+3)(n+4)}$$

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Hence, LHS. $= \sum_{r=0}^{n} \left({}^{(n+4)}C_{(r+4)} \cdot 3^{(r+4)} \right) \Big/ (n+1)(n+2)(n+3)(n+4)$

$$= \frac{\left((3+1)^{(n+4)} - {}^{(n+4)}C_0 \cdot 3^0 - {}^{(n+4)}C_1 \cdot 3^1 - {}^{(n+4)}C_2 \cdot 3^2 - {}^{(n+4)}C_3 \cdot 3^3\right)}{(n+1)(n+2)(n+3)(n+4)}$$

$= \text{RHS}$

$\Rightarrow \boxed{\text{LHS} = \text{RHS}}$

Q) find coeff of $x^{(n-2)}$ in $(x-1)(x-2)\dots(x-n)$.

A) Req $= \begin{array}{|l|} 1\cdot 2 + 1\cdot 3 + \dots + 1\cdot n \\ \quad + 2\cdot 3 + \dots + 2\cdot n \\ \quad\quad \vdots \quad\quad \vdots \\ \quad\quad\quad + (n-1)\cdot n \end{array}$

Observe,

$$\begin{array}{c|cccc} & 1 & 2 & \dots & n \\ \hline 1 & 1\cdot 1 & 1\cdot 2 & \dots & 1\cdot n \\ 2 & 2\cdot 1 & 2\cdot 2 & \dots & 2\cdot n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n\cdot 1 & n\cdot 2 & \dots & n\cdot n \end{array}$$

(Req.)

(Req.)

$2(\text{Req.}) + (1^2 + 2^2 + \dots + n^2) = (1+2+\dots+n)^2$

$$\Rightarrow \boxed{\text{Req.} = \left(\frac{1}{2}\right)\left[\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(2n+1)(n+1)}{6}\right]}$$

Q) If $(1+x)^n = C_0 + C_1x + \dots + C_nx^n$, then p.t.

1) $$\sum\sum_{0\le i<j\le n}\left(C_iC_j\right) = \left(2^{(2n-1)} - \frac{|\underline{2n}}{2\,|\underline{n}\cdot|\underline{n}}\right)$$

A)

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & n \\ \hline 0 & C_0C_0 & C_0C_1 & C_0C_2 & \dots & C_0C_n \\ 1 & C_1C_0 & C_1C_1 & C_1C_2 & \dots & C_1C_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n & C_nC_0 & C_nC_1 & C_nC_2 & \dots & C_nC_n \end{array}$$

(Req.)

(Req.)

$$2(\text{Req.}) + \left(C_0^2 + C_1^2 + \dots + C_n^2\right) = \left(C_0 + C_1 + \dots + C_n\right)^2$$

$$\Rightarrow \boxed{\text{Req.} = 2^{(2n-1)} - \frac{1}{2}\cdot {}^{2n}C_n}$$

2) $$\sum\sum_{0\le i<j\le n}\left((C_i+C_j)^2\right) = (n-1)\left({}^{2n}C_n\right) + 2^{2n}$$

A) $$2(\text{Req.}) + \left[(2C_0)^2 + (2C_1)^2 + \dots + (2C_n)^2\right] = \sum_{i=0}^{n}\sum_{j=0}^{n}\left((C_i+C_j)^2\right)$$

$$\Rightarrow \left[2(\text{Req.}) + 4\left({}^{2n}C_n\right)\right] = \sum_{i=0}^{n}\left(\overset{(n+1)}{C_i^2} + 2C_i\sum_{j=0}^{n}C_j + \sum_{j=0}^{n}C_j^2\right)$$

$$= \sum_{i=0}^{n}\left(\underset{(n+1)}{C_i^2} + 2^{(n+1)}C_i + {}^{2n}C_n\right)$$

$$\Rightarrow \left[2(\text{Req.}) + 4\left({}^{2n}C_n\right)\right] = {}^{2n}C_n(n+1) + 2^{(n+1)} \cdot 2^n + {}^{2n}C_n (n+1)$$

$$\Rightarrow \text{Req.} = \left[\frac{(2n+2-4)\left({}^{2n}C_n\right) + 2^{(2n+1)}}{2}\right]$$

$$\Rightarrow \boxed{\text{Req.} = (n-1)\left({}^{2n}C_n\right) + 2^{2n}}$$

3) $$\sum\sum_{0 \le i < j \le n} \left(C_i + C_j\right) = n \cdot 2^n$$

A) $$2(\text{Req.}) + \left[(2C_0) + (2C_1) + \ldots + (2C_n)\right] = \sum_{i=0}^{n}\sum_{j=0}^{n} \left(C_i + C_j\right)$$

$$\Rightarrow 2(\text{Req.}) + 2^{(n+1)} = \sum_{i=0}^{n} \left((n+1) C_i + 2^n\right)$$

$$= (n+1)2^n + (n+1)2^n \Rightarrow \text{Req.} = (n+1) \cdot 2^n - 2^n$$

$$\Rightarrow \boxed{\text{Req.} = n \cdot 2^n}$$

☆ Q) P.t. $$\sum\sum_{0 \le i < j \le n} \left((i+j) C_i C_j\right) = n\left[2^{(2n-1)} - \frac{1}{2} \cdot {}^{2n}C_n\right]$$

A) Observe if $i = (n - I)$ & $j = (n - J)$

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$$\Rightarrow \text{Req.} = \sum_{0\leq(n-I)<(n-J)\leq n}\sum \left((2n-I-J)\, C_{(n-I)}C_{(n-J)}\right)$$

$$= \sum_{0\leq J<I\leq n}\sum \left((2n-I-J)\, C_I C_J\right)$$

Since I & J are dummy variables,
$I \to j$, $J \to i$

$$\text{Req} = \sum_{0\leq i<j\leq n}\sum \left((2n-i-j)\, C_i C_j\right)$$

$$= (2n)\sum_{0\leq i<j\leq n}\sum (C_i C_j) - \sum_{0\leq i<j\leq n}\sum (i+j) C_i C_j$$

$$\Rightarrow \text{Req} = n \cdot \sum_{0\leq i<j\leq n}\sum (C_i C_j)$$

$$\boxed{\Rightarrow \text{Req.} = (n)\left[\frac{2^{(2n-1)} - {}^{2n}C_n}{2}\right]}$$

Q) $\displaystyle\sum_{0\leq i<j\leq n}\sum \left((i+j)(C_i + C_j + C_i C_j)\right)$

A) $i \to (n-j)$, $j \to (n-i)$, $\quad \text{Req.} = \displaystyle\sum_{0\leq i<j\leq n}\sum \left((2n-(i+j))(C_i + C_j + C_iC_j)\right)$

$$\Rightarrow \quad Req = \left(\frac{\cancel{2}n}{\cancel{2}}\right)\left(\sum\sum_{0\leq i<j\leq n}\left(C_i + C_j + C_iC_j\right)\right)$$

$$= (n)\left[\sum\sum_{0\leq i<j\leq n}(C_i+C_j) + \sum\sum_{0\leq i<j\leq n}(C_iC_j)\right]$$

$$= (n)\left[n\cdot 2^n + 2^{(2n-1)} - \frac{1}{2}\cdot {}^{2n}C_n\right]$$

$$\Rightarrow \quad \boxed{Req = \left(\frac{n}{2}\right)\left[2n\cdot 2^n + 2^{2n} - {}^{2n}C_n\right]}$$

Q) $$\sum\sum_{0\leq i<j\leq n}\left(i\,j\cdot C_i\cdot C_j\right) = (n^2)\left[2^{(2n-3)} - \frac{1}{2}\,{}^{(2n-2)}C_{(n-1)}\right]$$

A) $$i\cdot C_i = i\cdot\frac{\lfloor n}{\lfloor i\,\lfloor n-i} = n\cdot{}^{(n-1)}C_{(i-1)}$$

$$j\cdot C_j = j\cdot\frac{\lfloor n}{\lfloor j\,\lfloor n-j} = n\cdot{}^{(n-1)}C_{(j-1)}$$

$$\boxed{Req} = \sum\sum_{1\leq i<j\leq n}\left(n^2\cdot{}^{(n-1)}C_{(i-1)}\,{}^{(n-1)}C_{(j-1)}\right)$$

$$= (n^2)\left[2^{(2(n-1)-1)} - \frac{1}{2}\,{}^{2(n-1)}C_{(n-1)}\right] = \boxed{(n^2)\left[2^{(2n-3)} - \frac{1}{2}\,{}^{(2n-2)}C_{(n-1)}\right]}$$

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☆ Q) If $\sum_{r=0}^{2n}\left(a_r(x-2)^r\right) = \sum_{r=0}^{2n}\left(b_r(x-3)^r\right)$

and $a_k = 1,\ \forall k \geq n$. Show that $b_n = {}^{(2n+1)}C_{(n+1)}$

A) Let $y = (x-3)$.

$$\Rightarrow \sum_{r=0}^{2n}\left(b_r y^r\right) = \sum_{r=0}^{2n}\left(a_r (y+1)^r\right)$$

$$= \sum_{r=0}^{(n-1)}\left(a_r(y+1)^r\right) + \left[a_n(y+1)^n + a_{(n+1)}(y+1)^{(n+1)} + \ldots + a_{2n}(y+1)^{2n}\right]$$

$$= \sum_{r=0}^{(n-1)}\left(a_r(y+1)^r\right) + \left[(y+1)^n + \ldots + (y+1)^{2n}\right]$$

Coeff. of y^n in LHS $= b_n$

Coeff. of y^n in RHS $= \left({}^nC_n + {}^{(n+1)}C_n + {}^{(n+2)}C_n + \ldots + {}^{2n}C_n\right)$

Now, ${}^nC_n + {}^{(n+1)}C_n = {}^{(n+1)}C_{(n+1)} + {}^{(n+1)}C_n = {}^{(n+2)}C_{(n+1)}$

☆

$$\Rightarrow \left[{}^{(n+1)}C_{(n+1)} + {}^{(n+1)}C_n + {}^{(n+2)}C_n + \ldots + {}^{2n}C_n\right] = \left[{}^{(n+2)}C_{(n+1)} + {}^{(n+2)}C_n + \ldots + {}^{2n}C_n\right]$$

$$= \left[{}^{(n+3)}C_{(n+1)} + {}^{(n+3)}C_n + \ldots + {}^{2n}C_n\right] = \ldots = \left[{}^{(2n)}C_{(n+1)} + {}^{(2n)}C_n\right]$$

$$\Rightarrow \boxed{b_n = {}^{(2n+1)}C_{(n+1)}}$$

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<u>Alternate</u> — We know

$$(\text{Coeff. of } y^n) = b_n = \left(\text{Coeff. of } y^n \text{ in } \left[(1+y)^n + (1+y)^{(n+1)} + \ldots + (1+y)^{2n}\right]\right)$$

$$\Rightarrow \quad b_n = \left(\text{Coeff. of } y^n \text{ in } (1+y)^n \left[1 + (1+y) + \ldots + (1+y)^n\right]\right)$$

$$\Rightarrow \quad b_n = \left(\text{Coeff. of } y^n \text{ in } (1+y)^n \left(\frac{(1+y)^{(n+1)} - 1}{(1+y) - 1}\right)\right)$$

$$\Rightarrow \quad b_n = \left(\text{Coeff. of } y^{(n+1)} \text{ in } \left[(1+y)^{(2n+1)} - (1+y)^n\right]\right)$$

$$\Rightarrow \quad \boxed{b_n = {}^{(2n+1)}C_{(n+1)}}$$